Stability of attractors formed by inertial particles in open chaotic flows

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(Received 19 April 2004; published 13 September 2004)

Particles having finite mass and size advected in open chaotic flows can form attractors behind structures. Depending on the system parameters, the attractors can be chaotic or nonchaotic. But, how robust are these attractors? In particular, will small, random perturbations destroy the attractors? Here, we address this question by utilizing a prototype flow system: a cylinder in a two-dimensional incompressible flow, behind which the von Kármán vortex street forms. We find that attractors formed by inertial particles behind the cylinder are fragile in that they can be destroyed by small, additive noise. However, the resulting chaotic transient can be superpersistent in the sense that its lifetime obeys an exponential-like scaling law with the noise amplitude, where the exponent in the exponential dependence can be large for small noise. This happens regardless of the nature of the original attractor, chaotic or nonchaotic. We present numerical evidence and a theory to explain this phenomenon. Our finding makes direct experimental observation of superpersistent chaotic transients feasible and it also has implications for problems of current concern such as the transport and trapping of chemically or biologically active particles in large-scale flows.

DOI: 10.1103/PhysRevE.70.036203 PACS number(s): 05.45.−a, 05.40.−a

I. INTRODUCTION

The advective dynamics of idealized particles in two-dimensional, incompressible flows can be described as Hamiltonian [1,2]. In particular, consider such a flow characterized by a stream function $\Psi(x,y,t)$. For a particle with zero mass and size, its trajectory in the flow obeys the following equations:

$$\frac{dx}{dt} = \frac{\partial\Psi(x,y,t)}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial\Psi(x,y,t)}{\partial x}, \tag{1}$$

which are the standard Hamilton's equations of motion generated by the Hamiltonian $H(x,y,t)=\Psi(x,y,t)$. That is, the particle velocity $\mathbf{v}(x,y,t)=(dx/dt,dy/dt)$ follows exactly the flow velocity $\mathbf{u}(x,y,t)$, as given by the right-hand side of Eq. (1). This idealized picture changes completely when particles have finite mass and size. In this realistic case, the particle velocity is generally not the same as the flow velocity, and the equations of motion are no longer Hamilton's equations. Maxley and Riley were the first to consider this problem [3] by deriving a set of equations for the particle velocity $\mathbf{v}(x,y,t)$, taking into account physical effects due to finite mass and size such as the buoyancy force, the Stokes drag, the added inertia effect, and other corrections [3–5]. The resulting dynamical system is no longer Hamiltonian but dissipative instead. As such, attractors can arise [5–9]. Considering that, in an open Hamiltonian flow, ideal particles coming from the upper stream must necessarily go out of the region of interest in a finite amount of time, the formation of attractors of inertial particles is remarkable. Suppose these physical particles are biologically or chemically active. That they can be trapped permanently in some region in the physical space is of great interest or concern. A natural question is whether such attractors are structurally stable, i.e., whether they can persist under random perturbations.

In this paper, we are interested in the stability of attractors of inertial particles in open flows whose corresponding Lagrangian dynamics is chaotic. That is, for idealized particles, the equations of motion that they obey, namely Eq. (1), can exhibit chaos. We shall use the model of a two-dimensional flow passing a cylindrical obstacle, which was originally developed by Jung *et al.* [10]. The Reynolds number of the fluid and the geometrical parameters of the system are chosen such that behind the cylinder, chaotic vortices (von Kármán vortex street) form. Recent work by Benczik *et al.* [9] showed that for inertial particles in such a flow, attractors can be formed in the regions immediately behind the cylinder, but the attractors are usually not "stuck" on the cylinder. We call such attractors *inertial attractors*. By varying a system parameter, periodic and chaotic attractors, and in fact a complete cascade of period-doubling bifurcations to chaos, can be observed. As Benczik *et al.* pointed out, this result has implications in environmental science where forecasting aerosol and pollutant transport is a basic task, or even in defense applications where the spill of a toxin or biological pathogen in large-scale flows is of critical concern. The focus of this paper will be on the effect of noise on inertial attractors in this von Kármán vortex-street flow model. To make numerical computations and analysis feasible, we shall restrict our study to additive noise that is bounded in physically meaningful time. That is, we will consider Gaussian noise (white), but we restrict quantities of interest to time far less than the time to observe a rare, large amplitude realization of the Gaussian noise [11]. Under these considerations a noise amplitude can be meaningfully defined.

Our result is that under small noise, inertial attractors are typically destroyed, leaving behind a transient. If the original attractor is chaotic, the transient is chaotic. However, we find that even if the attractor is nonchaotic, under noise the resulting transient can still be chaotic. In both cases, the average lifetime τ of the chaotic transient obeys the following scaling law with noise amplitude ε:

$$\tau \sim \exp(C\varepsilon^{-\gamma}), \quad \text{for } \varepsilon > \varepsilon_c, \tag{2}$$

where $C > 0$ and $\gamma > 0$ are constants, and ε_c is related to the minimum distance between the attractor and its basin boundary [12]. The scaling law (2) is numerically observed to hold for close to two orders of magnitude of the noise variation. Here, τ is measured in terms of the natural time scale of the underlying flow, i.e., the cycle time of the vortices in the von Kármán street. For $\varepsilon \gg \varepsilon_c$, the average transient time is short. As ε is decreased the transient time increases following the scaling law (2) and, for $\varepsilon \sim \varepsilon_c$, the time can be as long as 1.5×10^5 flow cycles. The scaling law holds, regardless of whether the original attractor is chaotic or periodic. To account for the scaling law, we develop a physical theory, modeling the noise-induced escaping dynamics of trajectories from attractor by a class of stochastic differential equations. We will show that a properly formulated first-passage time problem of the corresponding stochastic system, in combination with the chaotic nature of the attractor under noise, leads to the scaling law (2).

The scaling law (2) is characteristic of *superpersistent chaotic transients*. By definition, a superpersistent chaotic transient is defined by the following scaling law for its lifetime [13]:

$$\tau \sim \exp(\alpha |p - p_c|^{-\gamma}), \tag{3}$$

where p is a system parameter, $\alpha > 0$ and $\gamma > 0$ are constants. We see that as p approaches the critical value p_c, the transient lifetime τ becomes superpersistent in the sense that the exponent in the exponential dependence diverges. This type of chaotic transient was discovered by Grebogi *et al.* [13]. They showed that the transients can generically occur through the dynamical mechanism of unstable-unstable pair bifurcation, in which an unstable periodic orbit in the boundary of a chaotic invariant set coalesces with another unstable periodic orbit pre-existing outside the set. The transients were also identified in a class of coupled-map lattices, leading to the speculation that asymptotic attractors may not be relevant for turbulence [14]. The unstable-unstable pair bifurcation mechanism was later shown to be responsible for the riddling bifurcation [15] that creates a riddled basin [16] in dynamical systems with symmetry. The mathematical models in Refs. [13,15] used to analyze superpersistent chaotic transients were discrete-time maps. Recently, superpersistent chaotic transients were demonstrated in systems described by differential equations, making *indirect* experimental observation of the transients possible [17]. In all these works, the transients occur in the *phase spaces* of dynamical systems. Our finding of superpersistent chaotic transients in a two-dimensional fluid system means that it may be possible to observe these transients directly, as for such a system the phase space and the physical space coincide. A short account of the work emphasizing this physical-space aspect of superpersistent chaotic transients has been published recently [18].

The rest of the paper is organized as follows. In Sec. II, we describe the flow model and the corresponding dissipative dynamical system that governs the motion of inertial particles. In Sec. III, we present numerical results of superpersistent chaotic transients for the two cases where the original attractor is chaotic and periodic, respectively. Section IV presents a theory to account for the numerically observed scaling laws for the average lifetime of the chaotic transients. Finally, a brief summary and a discussion of potential experimental systems for direct observation of superpersistent chaotic transients are offered in Sec. V.

II. MODEL OF ADVECTIVE DYNAMICS OF INERTIAL PARTICLES

A. Open-flow model

We use the open-flow model of the von Kármán vortex street in the wake of a cylinder, as detailed in Ref. [10]. The cylinder has radius r and is located in the middle of an infinite channel of width $4r$. The center of the cylinder is chosen to be the origin of the two-dimensional plane: $(x,y)=(0,0)$. The flow is incompressible and the Reynolds number is $R_e \equiv UL/\nu$, where U is a typical large-scale velocity and ν is the kinematic viscosity of the fluid. For not-too-small Reynolds numbers (say, $R_e \approx 250$ as in Ref. [10]), vortices form in the wake of the cylinder. The motions of the vortices in a background flow of velocity u_0 can be described by a time-periodic stream function $\Psi(x,y,t)$ (period $T_f=1$ in a standard dimensionless form), which was written heuristically to qualitatively describe the chaotic motion of Lagrangian passive tracers in the flow. The flow velocity $\mathbf{u}(x,y,t)$ can be obtained from $\Psi(x,y,t)$ according to the Hamilton's equations (1). By measuring the length in units of the cylinder radius r, which is also the characteristic linear size of the flow, the dimensionless model stream function can be written as

$$\Psi = fg,$$

where

$$f = f(x,y) = 1 - \exp[-(\sqrt{x^2+y^2} - 1)^2]$$

ensures the presence of a boundary layer

$$g = g(x,y,t) = -wh_1(t)g_1(x,y,t) + wh_2(t)g_2(x,y,t) + u_0 y_0 s(x,y)$$

describes the periodic detachment of the vortices, w represents the average strength of vortices

$$h_1(t) = \sin^2(\pi t),$$

$$h_2(t) = \cos^2(\pi t),$$

are functions characterizing the time evolution of the vorticity, u_0 is the dimensionless background velocity, and

$$s(x,y) = 1 - \exp(-(x-1)^2/c^2 - y^2)$$

is a shielding factor suppressing the background velocity in the wake. The factors

$$g_1(x,y,t) = \exp\{-R_0[(x-x_1(t))^2 + c^2(y-y_0)^2]\},$$

$$g_2(x,y,t) = \exp\{-R_0[(x-x_2(t))^2 + c^2(y+y_0)^2]\},$$

describe the Gaussian forms of the vortices of dimensionless size, whose positions in the wake are $[x_1(t), y_0]$ and $[x_2(t), -y_0]$. The functions x_1 and x_2 are defined by

$$x_1(t) = 1 + L_0 \bmod(t,1),$$

$$x_2(t) = 1 + L_0 \bmod\left(t - \tfrac{1}{2}, 1\right),$$

where L_0 is the dimensionless distance a vortex passes during its lifetime. In our numerical simulations we use the set of parameters in Refs. [9,10]: $u_0=14.0$, $R_0=0.35$, $y_0=0.3$, and $L_0=2.0$.

B. Inertial dynamics

For particles of finite mass and size advected in an incompressible flow, viscous friction arises and, as a result, the particle velocities differ from those of the fluid. Consider a spherical particle of radius a and mass m_p, and fluid of dynamic viscosity μ and element mass m_f. The equation of motion for the advective particle derived by Maxley and Riley [3] is

$$m_p \frac{d\mathbf{v}}{dt} = m_f \frac{d\mathbf{u}}{dt} - \frac{m_f}{2}\left(\frac{d\mathbf{v}}{dt} - \frac{d\mathbf{u}}{dt}\right) - 6\pi a \mu(\mathbf{v}-\mathbf{u}),$$

where on the right-hand side, the first term is the force from the undisturbed fluid flow, the second term is the force due to the added mass effect, and the third represents the Stokes drag. While in principle the fluid velocity $\mathbf{u}$ is disturbed by the particle motion, if the particle sizes are relatively small and their concentration is low, $\mathbf{u}$ can be considered as unchanged [3]. For convenience, one can introduce the mass ratio parameter

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \tag{4}$$

and the inertial parameter

$$A = \frac{R}{(2/9)(a/L)^2 R_e}, \tag{5}$$

where ρ_p and ρ_f are the densities of the particle and the fluid, respectively, and L is a typical large-scale mixing length. The equation of motion can then be cast in a dimensionless form

$$\frac{d\mathbf{v}}{dt} - \frac{3R}{2}\frac{d\mathbf{u}}{dt} = -A(\mathbf{v}-\mathbf{u}). \tag{6}$$

Inertial particles are *aerosols* if $0<R<2/3$, and they are *bubbles* if $2/3<R<2$. For the aerosol (bubble) case, the particle is heavier (lighter) than the surrounding fluid. The limit $A\to\infty$ corresponds to the situation of ideal particles (passive advection). The contraction rate of Eq. (6) is not zero so that the dynamical system as given by Eq. (6) is dissipative, allowing attractors to exist in the phase space. Although the attractors or other dynamical invariant sets of the system lie in the full phase space, they can be observed in the configuration or physical space [(x,y) plane].

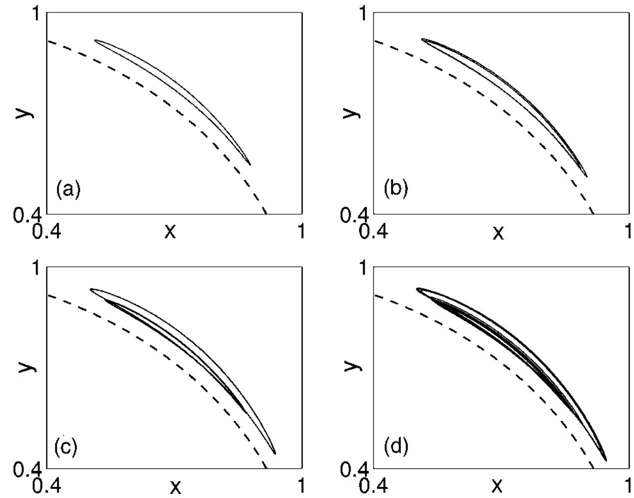

FIG. 1. Attractors in the configuration space. Periodic attractors with period (a) $T_p=1$ for $R=1.70$; (b) $T_p=2$ for $R=1.60$; and (c) $T_p=4$ for $R=1.50$. (d) Chaotic attractor for $R=1.47$. The dashed line indicates the cylinder in the upper half plane.

C. Attractors

It was demonstrated in Ref. [9] that attractors can be formed in the bubble regime, indicating that inertial particles can be trapped forever in the wake of the cylinder. We shall focus on the bubble regime by choosing the mass-ratio parameter from the range $2/3<R<2$. For concreteness, we fix the inertial parameter at $A=30$. Typically, there are three attractors [9]: two located symmetrically with respect to the x axis near the cylinder but not stuck on it, and the third one at $x=+\infty$ (for flows from $x=-\infty$). Some representative examples of the attractors are displayed in Figs. 1(a)–1(d). For instance, for $R=1.70$ we observe limit-cycle attractors, as shown in the $y>0$ plane in Fig. 1(a). As R decreases, a cascade of period-doubling bifurcations occurs, leading to chaotic attractors for $R\lesssim 1.475$. See Fig. 2.

The largest Lyapunov exponents of the attractors can be conveniently calculated by using the standard time-series method [19]. This is particularly useful for experimental situations where the equations of motion are not available, or for sophisticated numerical models for which the evaluation of the Jacobian matrices is difficult. Figure 3 shows, for $R=1.47$, a histogram of the largest exponent estimated from 250 trajectories in finite time (1000 flow cycles). The center of the distribution is $\lambda_1\approx 0.55$, which is a good approximation of the largest Lyapunov exponent of the chaotic attractor.

III. NOISE-INDUCED SUPERPERSISTENT CHAOTIC TRANSIENTS

To simulate random forcing due to the flow disturbance or other environmental factors, we add terms $\varepsilon\xi_x(t)$ and $\varepsilon\xi_y(t)$ to the force components in the x- and y-directions, where

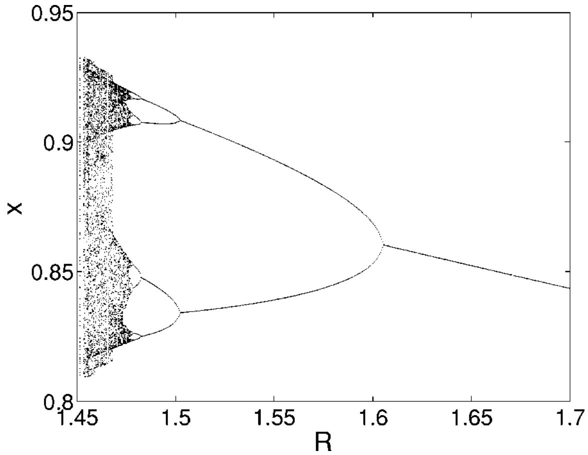

FIG. 2. A typical bifurcation diagram as a function of the mass-ratio parameter R. A complete cascade of period-doubling bifurcations to chaos is present.

$\xi_x(t)$ and $\xi_y(t)$ are independent Gaussian random variables of zero mean and unit variance, and ε is the noise amplitude. We stress that we focus on time scales that are much shorter than the time to observe any large amplitude events from the Gaussian distribution. The equations of motion under noise are

$$\frac{dx}{dt} = v_x,$$

$$\frac{dy}{dt} = v_y, \tag{7}$$

$$\frac{d\mathbf{v}_x}{dt} = \frac{3R}{2}\frac{d\mathbf{u}_x}{dt} - A(\mathbf{v}_x - \mathbf{u}_x) + \varepsilon \xi_x(t),$$

$$\frac{d\mathbf{v}_y}{dt} = \frac{3R}{2}\frac{d\mathbf{u}_y}{dt} - A(\mathbf{v}_y - \mathbf{u}_y) + \varepsilon \xi_y(t).$$

A standard second-order method [20] was utilized to integrate the set of stochastic differential equations (7). In what

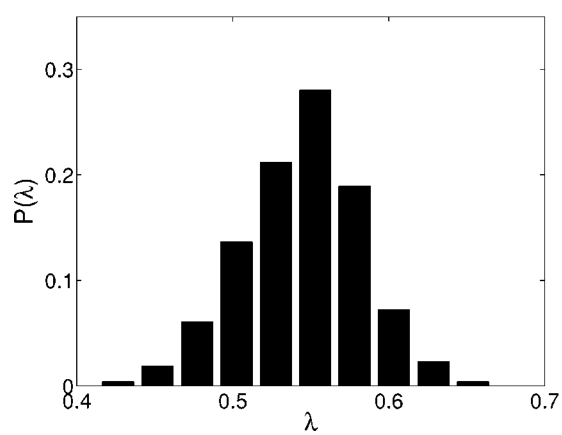

FIG. 3. Histogram of the largest Lyapunov exponent of the chaotic attractor for $R=1.47$, estimated in finite time.

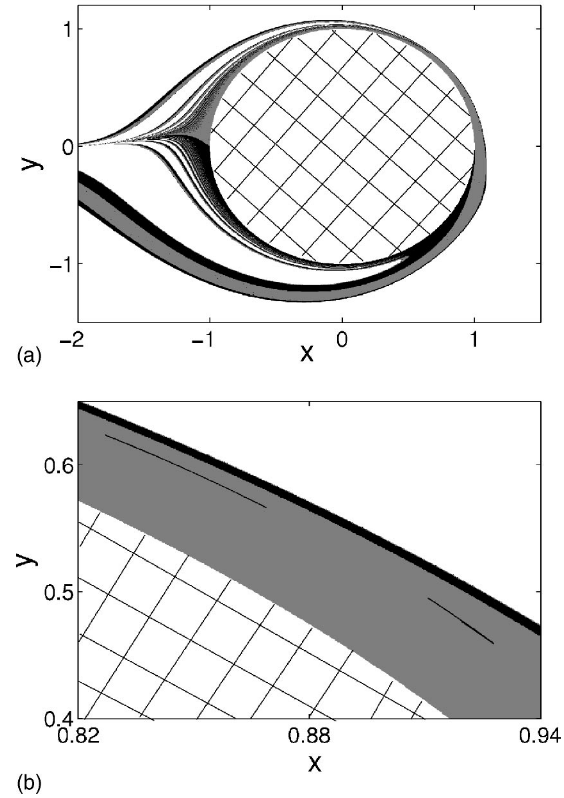

FIG. 4. (a) Basins of attraction of two chaotic attractors (gray and black, respectively) in the absence of noise. The white region denotes the basin of the attractor at $x=\infty$ and the meshed region the cylinder. (b) Upper chaotic attractor (black in gray), which appears to be close to the basin boundary.

follows we present numerical evidence for noise-induced superpersistent chaotic transients for the two cases where the deterministic attractor is chaotic and periodic, respectively.

A. Chaotic attractors

To gain insight as to what might happen to the chaotic attractors under noise, we examine the basins of attraction of these attractors at $R=1.47$. To do so we choose a 1000 $\times$1000 grid of initial conditions in the region $[-2.0 \leqslant x(t_0) \leqslant 1.5, -1.5 \leqslant y(t_0) \leqslant 1.2]$ covering the cylinder and $t_0=0.2$, and set the initial velocities to be $v_x(t_0)=u_x(x,y,t_0)$ and $v_y(t_0)=u_y(x,y,t_0)$, and then compute toward which attractor every initial particle is attracted. Figure 4(a) shows, on a stroboscopic section in the two-dimension configuration space, the basins of attraction of the two chaotic attractors (gray and black, respectively), where the blue region denotes the basin of the attractor at infinity. Figure 4(b) is a blowup of part of the basin that contains the chaotic attractor in the upper half plane. It is apparent that the attractor is close to the basin boundary.

Note that the phase space is five-dimensional, so what is shown in Figs. 4(a) and 4(b) is in fact a two-dimensional slice of the basin structure in the full phase space, which corresponds to the physical space. Near the cylinder, the basin boundaries among the three attractors are apparently frac-

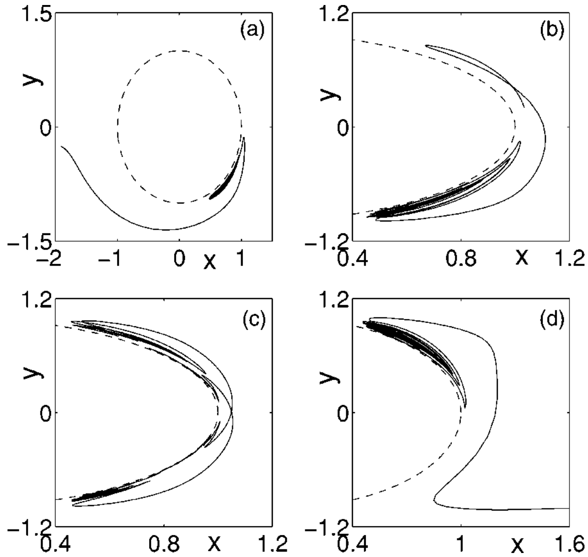

FIG. 5. For $R=1.47$ and $\varepsilon=\exp(-2.2)$, an escaping chaotic trajectory in time sequence [(a)→(d)]. The cylinder is denoted by the dashed line.

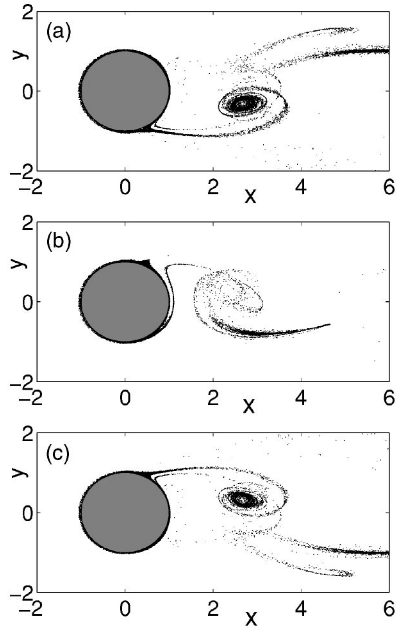

FIG. 6. (a)–(c) At three different instants of time, $T_f/4$ apart, locations of the temporally trapped and escaping particles in the physical space.

tal [21]. Because of the explicit time dependence in the stream function and therefore in the flow velocities, the attractors and their basins oscillate around the cylinder. The remarkable feature is that in the physical space, there are time intervals during which the attractors come close to the basin boundaries. Thus, under noise, we expect permanently trapped motion on any one of the two chaotic attractors to become impossible. In particular, particles can be trapped near the cylinder, but this can last only for a finite amount of time: eventually, all trajectories on these attractors escape and approach the $x=\infty$ attractor. That is, chaos becomes transient under noise.

We have seen that due to the symmetry of the flow with respect to the x axis, there are two attractors located symmetrically in the upper- and lower-half planes, respectively. Because of the convoluted basins of attraction of these attractors [Fig. 4(a)], under noise of proper amplitude we expect to see a switching behavior in a typical trajectory between the two original attractors, before it escapes to the attractor at $x=+\infty$. Figures 5(a)–5(d) show such a trajectory in four intervals of time for $\varepsilon=\exp(-2.2)$. Specifically, in Fig. 5(a), the trajectory comes from the upper stream of the flow (to the left-hand side of the cylinder) and gets attracted to the original attractor in the lower half plane. In Fig. 5(b), the trajectory begins to switch to the original attractor in the upper half plane. The switching behavior can be seen more clearly in Fig. 5(c). Eventually, the trajectory escapes to the infinity attractor along the down stream of the flow, as shown in Fig. 5(d).

To understand the nature of the noise-induced transient chaos, we distribute a large number of particles in the original basins of the chaotic attractors, and examine the channel(s) through which they escape to the $x=+\infty$ attractor under noise. Figures 6(a)–6(c) show, for three instants of time (t, $t+T_f/4$, and $t+T_f/2$, respectively), locations of an ensemble of particles in the physical space. Due to the symmetry of the flow [10], the particle trajectories at t and $t+T_f/2$ are symmetric to each other with respect to the x axis, as can be seen from Figs. 6(a) and 6(c). While there are particles still trapped in the original attractors, many others are already away from the cylinder. Since this is a two-dimensional projection of a five-dimensional dynamics, some fractal-like features overlap. The channels through which they escape are a set of thin openings surrounding the cylinder and extending to one of the vortices in the flow. After wandering near the vortex, particles go to the $x=+\infty$ attractor. Because of the time-dependent nature of the flow, in the physical space the locations of these openings vary in time, but the feature that they are narrow is shared. For a fixed noise amplitude, numerically we find that the lifetimes of the particles near the cylinder obey an extremely slow, exponentially decaying distribution, from which the average lifetime τ is obtained. Figures 7(a) and 7(b) show τ versus the noise amplitude ε on two different scales, where τ is measured in units of the period of the flow cycle. Note that for $\varepsilon=0$, there is an attracting motion so that τ diverges. Figure 7(b) suggests, however, the way that τ increases follows the superpersistent transient scaling law (2) as ε is decreased. The scaling is valid in a finite range of the noise amplitude. In particular, in order to observe the escape of particles in realistic time, the noise amplitude should be large enough for a trajectory on a chaotic attractor to cross its basin boundary. Roughly, to induce escape, the minimum noise amplitude ε_c required is proportional to the minimum distance between the attractor and its basin boundary. Numerically we find $\varepsilon_c \sim e^{-2.5}$, for which the transient lifetime is on the order of $e^{12} \sim 10^5$ flow cycles. The numerically obtained superpersistent transient scaling law holds in the range that spans over one order of magnitude of the noise variation above ε_c.

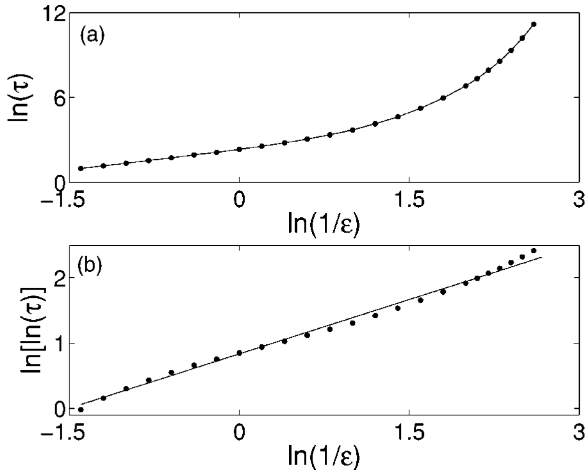

FIG. 7. Scaling of the average lifetime of the trapped chaotic particles versus the noise amplitude on two different scales. A linear plot in the double logarithmic versus logarithmic scale in (b) suggests that the chaotic transients are superpersistent.

B. Periodic attractors

When the deterministic attractor is periodic, we find that noise can also induce transient dynamics in that particles initially trapped in the attractor will eventually escape when noise amplitude is larger than a threshold value. To illustrate this behavior, we plot in Figs. 8(a)–8(d) a deterministic periodic attractor in the physical space for $R=1.6$, and an escaping, noisy trajectory in three intervals of time before it exits the wake region for $\varepsilon = e^{-2.1}$. In Fig. 8(b), the trajectory comes from the upper stream of the flow and is temporally trapped in the region where the original periodic attractor resides. Figure 8(c) shows the switching behavior between the two symmetric attractors due to noise, and Fig. 8(d)

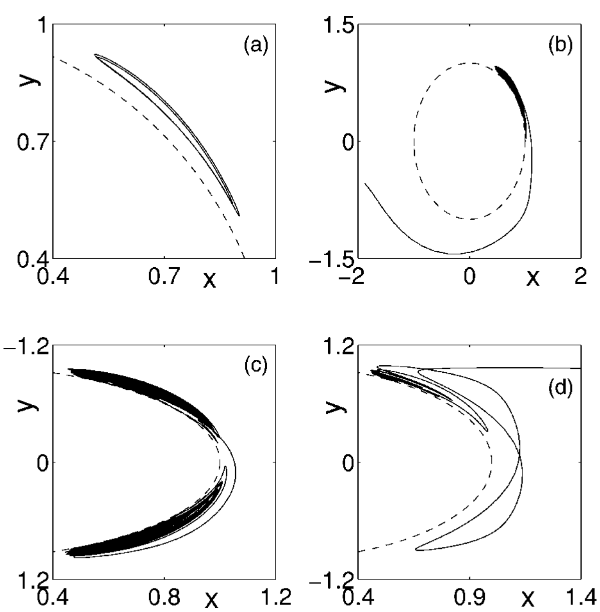

FIG. 8. For $R=1.60$ (a) periodic attractor in the absence of noise. (b),(c) For $\varepsilon = \exp(-2.1)$, behaviors of a typical trajectory near the original attractor before it exits downstream in the flow.

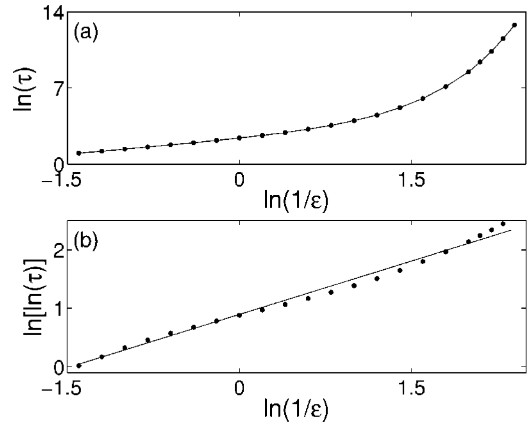

FIG. 9. For $R=1.60$, scaling of the average lifetime of temporally trapped trajectories near the periodic attractors with noise, plotted on two different scales. The transient behavior can apparently be classified as superpersistent.

shows the trajectory before it escapes. We notice that the behaviors are similar to those in Fig. 5 where the original deterministic attractor is chaotic. The interesting result is that the scaling of the average transient lifetime with the noise amplitude appears to be superpersistent as well, as shown in Fig. 9, where the approximately linear behavior in (b) indicates the scaling law (2) for over at least one order of magnitude of the noise variation.

To test whether the transient dynamics is chaotic, we compute the finite-time distribution of the largest Lyapunov exponent, as shown in Fig. 10(a), where the number of trajectories used is 250, the time is 1000 flow cycles, and the exponent is again estimated from the time-series method. We see that the distribution is on the positive side, with center at $\lambda_1 \approx 0.118$, indicating a chaotic transient. We also find that as the noise amplitude is increased, the center of the distribution increases as well, as shown in Fig. 10(b). Thus, in the parameter regime where the deterministic attractor is not chaotic, noise can still induce superpersistent chaotic transients.

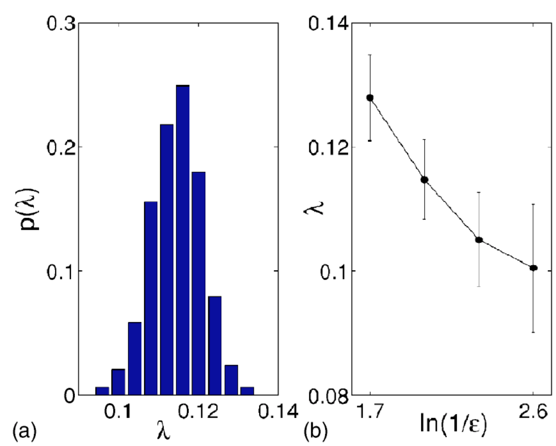

FIG. 10. (a) Finite-time distribution of the largest Lyapunov exponent for $\varepsilon = \exp(-2.0)$ and $R=1.6$ where the deterministic attractor is periodic. The distribution is on the positive side, indicating that the transient dynamics is chaotic. (b) Center of the distribution versus the noise amplitude.

IV. SCALING THEORY

We shall provide a physical theory for the scaling of the average lifetime of noise-induced superpersistent chaotic transients. Previous works suggest unstable-unstable pair bifurcation as the generic mechanism for the transients [13,15]. One can imagine two unstable periodic orbits of the same periods, one on the chaotic attractor and another on the basin boundary. In a noiseless situation, as a bifurcation parameter p passes through a critical value $p_c(p \gtrsim p_c)$, the two orbits *coalesce* and disappear simultaneously, leaving behind a set of narrow "channels" in the phase space through which trajectories on the chaotic attractor can escape. There is superpersistent chaotic transient for $p > p_c$. Our fluid problem corresponds to the parameter regime of $p \lesssim p_c$ (subcritical regime) because, in the absence of noise, there are attractors. In this case, noise can induce a set of channels that opens and closes stochastically in time, through which trajectories on the attractors can escape. In what follows, to gain insight we first describe the scaling of the average lifetime of superpersistent chaotic transient with parameter p in the deterministic case. We then develop a model based on a class of first-order stochastic differential equations to account for the scaling law governing noise-induced superpersistent chaotic transients.

A. Scaling law of superpersistent chaotic transients in the deterministic case

Grebogi, Ott, and Yorke [13] used the following class of two-dimensional, noninvertible maps on cylinder to first report superpersistent chaotic transients:

$$\theta_{n+1} = 2\theta_n \bmod 2\pi,$$
$$z_{n+1} = az_n + z_n^2 + \beta \cos \theta_n, \qquad (8)$$

where a and β are parameters. Because of the z_n^2 term in the z equation for large $|z_n|$ we have $z_{n+1} > z_n$. There is thus an attractor at $z = +\infty$. Near $z=0$, depending on the choice of the parameters, there can be either a chaotic attractor or none. For instance, for $0 < \beta \ll 1$, there is a chaotic attractor near $z=0$ for $a < a_c = 1-2\sqrt{\beta}$ and the attractor becomes a chaotic transient for $a > a_c$ [13]. The transient is superpersistent for $a > a_c$, which can be argued [13] as follows.

For $a < a_c$ there are two fixed points: $(\theta_1, z_1) = (0, z_b)$ and $(\theta_2, z_2) = (0, z_c)$, where $z_{c,b} = (1-a \pm r)/2$ and $r = \sqrt{(1-a)^2 - 4\beta}$. The fixed points $(0, z_b)$ and $(0, z_c)$ are on the basin boundary and on the chaotic attractor, respectively. They coalesce at $a = a_c$. For $a > a_c$, a channel is created through which trajectories on the original attractor can escape to an attractor at infinity. At the location of the channel where $\theta = 0$, the z mapping can be written as

$$z_{n+1} - z_n = (a-1)z_n + z_n^2 + \beta.$$

Letting $\delta = z - z_*$, where z_* is the minimum of the quadratic function of z on the right-hand side, we have

$$\delta_{n+1} = \delta_n + \delta_n^2 + b, \qquad (9)$$

where $b = \sqrt{\beta}(a-a_c) - [(a-a_c)/2]^2$. For $a > a_c$, we have $b \approx \sqrt{\beta}(a-a_c)$. In the continuous-time approximation, the dynamics in δ can be described by $d\delta/dt = \delta^2 + b$. Thus, the time T required to tunnel through the escaping channel is

$$T \approx \frac{1}{b^{1/2}} \int_0^\infty \frac{d\delta}{\delta^2 + 1} = \frac{\pi}{2b^{1/2}}.$$

Since the θ dynamics is uniformly chaotic with Lyapunov exponent $\lambda = \ln 2 > 0$, the probability for a trajectory to fall in the opening of the channel and to stay near there in the θ direction for consecutively T iterations is proportional to $e^{-T \ln 2}$. For $a > a_c$, the average chaotic transient time is thus given by

$$\tau \sim e^{T \ln 2} \approx e^{(\pi \ln 2/2) b^{-1/2}} \approx \exp[C(a - a_c)^{-1/2}], \qquad (10)$$

where $C = \pi(\ln 2)\beta^{-1/4}/2$ is a positive constant. Thus, in the noiseless situation, for $a > a_c$ the exponent in the scaling of the average tunneling time with the parameter variation assumes the value of $1/2$, which was verified numerically [13].

In the above example, the two periodic orbits involved in the unstable-unstable pair bifurcation that induces a superpersistent chaotic transient are a saddle and a repeller, which are allowed if the two-dimensional map is noninvertible. For invertible maps, the required phase-space dimension for the bifurcation is at least three (or at least four for flows). Thus, for most commonly studied low-dimensional chaotic systems such as two-dimensional invertible map or three-dimensional flows, unstable-unstable pair bifurcation and hence superpersistent chaotic transients cannot occur. For simplicity, to study the effect of noise, we will use two-dimensional noninvertible maps as prototype models.

B. Scaling law of noise-induced superpersistent chaotic transients

In our fluid-flow problem, transient chaos is induced by noise. As can be seen in Fig. 4(b), the existence of attractors and their closeness to the basin boundary in the absence of noise imply that the setting belongs to the subcritical case. Under noise, an escaping channel may open at the location of an unstable periodic orbit. For small noise, the probability for the channel to open is small. Particles can move, however, to the location of the channel and remain there for a finite amount of time to escape through the channel while it is open. Suppose, on average, it takes time $T(\varepsilon)$ for a particle to travel through the channel. We expect $T(\varepsilon)$ to increase as the noise amplitude ε is decreased, because the probability for the channel to remain open is smaller for weaker noise. In fact, as we will argue below, we expect $T(\varepsilon)$ to increase at least algebraically as ε is decreased to a critical noise amplitude ε_c. The analysis below assumes the existence of a chaotic attractor, but it also applies to the case where the attractor is periodic, insofar as noise can induce chaotic motion before escaping, as observed in our fluid problem.

Suppose the largest Lyapunov exponent of the chaotic attractor is $\lambda > 0$. After an unstable-unstable pair bifurcation the opened channel is locally transverse to the attractor. In order for a trajectory to escape, it must spend at least time $T(\varepsilon)$ at the location of the opening on the attractor. The trajectory must come to within a distance of about

$\exp[-\lambda T(\varepsilon)]$ from the location of the channel. The probability for this to occur is proportional to $\exp[-\lambda T(\varepsilon)]$. The average time for the trajectory to remain on the attractor, or the average transient lifetime, is thus $\tau \sim \exp[\lambda T(\varepsilon)]$.

To obtain the dependence of $T(\varepsilon)$ on ε, we consider a small region about the "root" of the channel, or the location of the mediating periodic orbit. Let $\mathbf{q}$ and $\mathbf{z}$ be the local coordinates on the attractor and in the channel, respectively. We consider the following model:

$$\frac{d\mathbf{q}}{dt} = \mathbf{f}(\mathbf{q}),$$

$$\frac{d\mathbf{z}}{dt} = \varepsilon\xi(t) + \mathbf{g}(\mathbf{q})p + \mathbf{h}(\mathbf{z}),$$

where p is a parameter, $\mathbf{f}(\mathbf{q})$ is a vector field that generates a chaotic attractor, $\varepsilon\xi(t)$ is noise, the lowest order of the function $\mathbf{h}(\mathbf{z})$ is $\mathbf{z}^2$. The function $\mathbf{g}(\mathbf{q})$ satisfies the condition $\|\mathbf{g}(\mathbf{q})\| < D$ for $\mathbf{q} \neq \mathbf{0}$, $\mathbf{g}(\mathbf{0}) = \mathbf{1}$, where D is a positive constant and $\|\cdot\|$ denotes a proper norm. For $\varepsilon = 0$ and $p \lesssim 0$, we have $\|g(\mathbf{q})\| p \lesssim \mathbf{0}$, so that the channel is closed and no trajectory can escape. For $p \gtrsim 0$, the probability for the channel to open is finite, and so is the probability for trajectories to escape. It is thus convenient to set $p_c = 0$, where $p \lesssim p_c$ corresponds to the subcritical case, as in our fluid problem.

In order to construct a model that captures the essential transient dynamics, while at the same time is amenable to analysis, we assume that the escaping channel is approximately one-dimensional and the length of the channel is $l \gg \varepsilon$. This picture can be justified for the typical case where the periodic orbit at the opening of the channel in the original attractor is strongly unstable in the direction of the channel so that the escaping dynamics in the channel is approximately one-dimensional. Once a trajectory on the chaotic attractor falls into the opening of the channel, i.e., $\mathbf{q} = \mathbf{0}$, its motion is governed by a stochastic differential equation of the form

$$\frac{dz}{dt} = h(z) + p + \varepsilon\xi(t).$$

The probability density function $\phi(z,t)$ of the corresponding stochastic process obeys the Fokker-Planck equation

$$\frac{\partial\phi}{\partial t} = -\frac{\partial}{\partial z}([h(z) + p]\phi) + \frac{\varepsilon^2}{2}\frac{\partial^2\phi}{\partial z^2},$$

where ε^2 is the diffusion coefficient. For small z we consider the lowest order of $h(z)$ and write $h(z) \approx az^{k-1}$, where $a > 0$ and $k \geqslant 3$. The time required for a trajectory to travel through the channel is roughly the mean first passage time [22]

$$T(\varepsilon) = \frac{2}{\varepsilon^2}\int_0^l dy \exp[-W(y)/\varepsilon^2]\int_0^y \exp[W(z)/\varepsilon^2]dz,$$

where $W(z) = H(z) + 2pz$ and $H(z) = \int 2h(z)dz$. Depending on the noise amplitude, the exponential terms inside the integral can be approximated as $\exp\{-[H(z) + 2pz]/\varepsilon^2\} \approx \exp[-H(z)/\varepsilon^2]$, if $\varepsilon \gg |p|^\sigma$, where $\sigma = k/(2k-2)$. Thus, we have

$$T(\varepsilon) \sim \frac{2}{\varepsilon^2}\int_0^l dy \exp[-H(y)/\varepsilon^2]\int_0^y \exp[H(z)/\varepsilon^2]dz.$$

Using the series expansion of the exponential term $\exp[H(z)/\varepsilon^2]$, $T(\varepsilon)$ can be evaluated (see the Appendix). We have

$$T \lesssim \varepsilon^{-2+4/k},$$

which, when substituted into $\tau \sim \exp[\lambda T(\varepsilon)]$, gives the scaling law (3) that is characteristic of superpersistent chaotic transients. We see that the exponent is $\gamma = 2 - 4/k$. In general, we have $0 < \gamma < 2$. In principle, the scaling law (3) holds only when the noise amplitude exceeds the critical value $\varepsilon_c \equiv |p|^\sigma$. For a realistic problem, the critical noise amplitude ε_c is related to the minimum distance between the attractor and its basin boundary in the deterministic case.

V. DISCUSSION

We have investigated the stability of attractors formed by inertial particles behind structures in a prototype open-flow system consisting of a cylindrical obstacle in an infinite channel. We consider the setting where fluid is incompressible and chaotic vortices form behind the cylinder. For ideal particles with zero mass and size, the corresponding dynamical system is open Hamiltonian so that particles from the upper stream of the flow can be in the vicinity of the cylinder only for a finite amount of time before exiting this region downstream. Recent work showed that for physical particles with finite inertia and size, attractors, chaotic or nonchaotic, can form behind the cylinder [9]. This raises concern for situations where, for instance, advective particles in the atmosphere are chemically or biologically active and can be trapped behind some structures. The result of this paper indicates that such attractors are not stable in the sense that they can be destroyed by small noise. However, the resulting transient can be extremely long in that its average lifetime obeys the scaling law that is characteristic of superpersistent chaotic transients. As a practical matter, the extraordinarily long trapping time makes the transient particle motion practically equivalent to an attracting motion with similar physical or biological effects. We wish to remark that, while there are numerous results in the mathematics literature concerning the persistence of attractors under noise [24,25], the issue addressed in this paper concerns *what can happen* physically to an attractor when it can *no longer* persist under random perturbations.

As a by-product, our result represents evidence of superpersistent chaotic transients in the physical or configuration space, whereas to our knowledge, most previous works on this type of chaotic transients focused on the phase space of dynamical systems [13–15,17]. It may then be possible to observe superpersistent chaotic transients directly in fluid experiments using open chaotic flows. While we recognize that to confirm a scaling law from experimental data can be extremely difficult in fluid dynamics (see, for example, the contradictory conclusions in Refs. [26,27] and on whether the scaling of velocity profiles in a turbulent boundary layer is Reynolds-number dependent), we think it is possible to experimentally observe the chaotic-transient scaling law (2). For instance, in the experimental system in Ref. [28], a wake is created in the flow passing a cylinder, and dyes are used to allow chaotic scattering dynamics to be traced. We can imagine using dyes of finite mass to generate attracting motions behind the cylinder. As perturbations of controllable magnitude (such as small, forced random vibration of the cylinder) are applied, the dyes trapped behind the cylinder will escape down the flow. By measuring the average time for which the dyes stay near the cylinder as a function of the magnitude of the perturbations, the scaling law (2) can be tested. Another possible experimental setting is the fluid mixing system in a stirred tank [29].

ACKNOWLEDGMENT

This work is supported by AFOSR under Grant No. F49620-03-1-0290.

APPENDIX

The series expansion

$$\exp[H(z)/\varepsilon^2] = \sum_{n=0}^{\infty} \frac{(bz^k)^n}{n!},$$

gives

$$\int_0^y \exp[H(z)/\varepsilon^2]dz = \sum_{n=0}^{\infty} \frac{b^n y^{kn+1}}{n\,!\,(kn+1)},$$

where $b=(2a)/(k\varepsilon^2)$. It implies that

$$T \sim \frac{2}{\varepsilon^2}\int_0^l \sum_{n=0}^{\infty} \frac{(by^k)^n}{n\,!\,(kn+1)} y \exp(-by^k)dy.$$

Let $q=by^k$, we obtain

$$T \approx \frac{2}{\varepsilon^2}\sum_{n=0}^{\infty} \frac{1}{n\,!\,(kn+1)}\int_0^{l'} q^n \exp(-q)\frac{1}{kb}\left(\frac{q}{b}\right)^{-(k-2)/k} dq$$

$$\sim \varepsilon^{-2+4/k}\sum_{n=0}^{\infty} \frac{1}{n\,!\,(kn+1)k}\int_0^{l'} q^{n-(k-2)/k} \exp(-q)dq,$$

$$\sim \varepsilon^{-2+4/k}\sum_{n=0}^{\infty} \frac{1}{n\,!\,(kn+1)k}\Gamma(n+2/k),$$

where $l'=bl^k$ and $\Gamma(x)=\int_0^\infty t^{x-1}\exp(-t)dt$ is the gamma function. In order to show that the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n\,!\,(kn+1)k}\Gamma(n+2/k) \tag{A1}$$

converges, we use the following property of the gamma function [23]:

$$\frac{b^{b-1}}{a^{a-1}}\exp(a-b) < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}\exp(a-b)(b > a \geqslant 1).$$

Using this double inequality and $\Gamma(n+1)=n!$, we obtain the upper bound of $\Gamma(n+2/k)$

$$\Gamma(n+2/k) < \Gamma(n+1)\exp(1-2/k)\frac{(n+2/k)^{n-1+2/k}}{(n+1)^n}$$

$$< n\,!\ \exp(1-2/k)(n+2/k)^{-1+2/k}.$$

Substituting this upper bound into the infinite series (A1), we can show that the series is convergent, as follows:

$$\sum_{n=0}^{\infty} \frac{1}{n\,!\,(kn+1)k}\Gamma(n+2/k)$$

$$< \sum_{n=0}^{\infty} \frac{1}{(kn+1)k}\exp(1-2/k)(n+2/k)^{-1+2/k}$$

$$< \frac{\exp(1-2/k)}{k^2}\sum_{n=0}^{\infty} \frac{1}{(n+1/k)^{2-2/k}} < \infty.$$

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